

Evolutionary minority game with multiple options

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We propose and study an evolutionary minority game (EMG) in which the agents are allowed to choose among three possible options. Unlike the original EMG where the agents either win or lose one unit of wealth, the present model assigns one unit of wealth to the winners in the least popular option, deducts one unit from the losers in the most popular option, and awards R ($-1 < R < 1$) units for those in the third option. Decisions are made based on the information in the most recent outcomes and on the characteristic probabilities of an agent to follow the predictions based on recent outcomes. Depending on R , the population shows a transition from self-segregation in difficult situations ($R < R_c$) in which the agents tend to follow extreme action to cautious or less decisive action for $R > R_c$, where $R_c(N)$ is a critical value for optimal performance of the system that drops to zero as the number of agents N increases.

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I. INTRODUCTION

Agent-based models of complex adaptive systems [1] have recently attracted much attention among scientists in different research areas [2–4]. Typically, these models consist of a competing population in which agents decide based on some global information. The information is, in turn, the result of the collective behavior of the population. An outstanding example of an agent-based model is that of the minority game (MG), which is a binary version of Arthur's bar attendance problem [5], proposed by Challet and Zhang [6]. In the MG, agents compete to be in a minority group and make decisions based on information generated by the actions of the agents in previous rounds without direct interaction among the agents. The population show interesting cooperative actions [7–10]. The MG forms the basis of many interesting agent-based models in recent years [11], and many of its variations were proposed in connection to possible applications in areas such as complex systems and econophysics [3]. These variations differ from the original MG mainly in the way in which the adaptive ability of the agents is introduced into the models.

In the evolutionary minority game (EMG) proposed by Johnson *et al.* [12–16], an odd number N of agents decide to choose one of two options, 0 or 1, at each time step. The agents who are in the minority (majority) group win (lose) and are awarded (deducted) one point. Every agent holds the same dynamical strategy of simply following the most recent trend, together with two individual parameters, namely a score and a probability p ($0 \leq p \leq 1$). The probability p is the chance that an agent decides to follow the strategy's prediction and $1-p$ is the chance that the agent decides to act opposite to the current trend. The p value of an agent is allowed to change, if the agent does not perform well. If the

score of an agent drops below a threshold d ($d \leq 0$), the agent replaces his p value by a new value randomly taken within a range r of the original p value and his score is reset to zero. The most interesting feature in EMG is that agents who behave in an extreme way (i.e., using $p \approx 0$ and $p \approx 1$) perform better than the cautious ones (i.e., using $p \approx 0.5$) [12,13]. This in turn leads to a self-segregation of the population in the sense that the distribution of p values tends to peak at $p \approx 0$ and $p \approx 1$. Hod and Nakar [17] studied a modified version of EMG with a biased payoff function, i.e., the points awarded to winners and deducted from losers are different. The authors found that self-segregation occurs only if the ratio \bar{R} of the point awarded to the point deducted is greater than or equal to some value \bar{R}_c . Note that \bar{R}_c depends on the size of the population. In the limit of a large population, $\bar{R}_c \rightarrow 1$. For $\bar{R} < \bar{R}_c$, i.e., corresponding to difficult situations in which winning in a turn cannot compensate for losing in another turn, the distribution of p values shows the feature of clustering by which agents tend to be more cautious or less decisive and take on p values around 0.5.

Motivated by real-life scenarios in which there may be more than one winning decision with different payoffs and by the different behavior of the modified EMG [17] for $\bar{R} \geq \bar{R}_c$ and $\bar{R} < \bar{R}_c$, we propose and study a generalized version of EMG with three possible options. The agents in the most (least) popular option are deducted (awarded) one point, while the agents in the third option are awarded R points, where R may be positive or negative with $|R| < 1$. Each agent has the probabilities $(p_1, p_2, p_3 = 1 - p_1 - p_2)$ to follow the prediction of the most current trend of the outcomes. It is found that a transition from self-segregation in the distribution of p values to cautious behavior occurs as R changes. At some critical value R_c , the system performs optimally. The plan of the paper is as follows. In Sec. II, the three-option EMG is defined. We present and discuss results of detailed numerical simulations in Sec. III. Results are summarized in Sec. IV.

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II. THREE-OPTION EVOLUTIONARY MINORITY GAME

The present model represents a generalization of the original EMG [12] from two options to three options. Our model is motivated by situations in which there may be more than one winning or losing options. Many agent-based models were proposed, for example, with possible applications related to the features observed in financial markets [3]. Taking the financial market as an example, agents may have many choices on whether to invest on the stock market, the money market, bonds, derivatives, etc., depending on the availability of funds, the level of risk that one may want to handle, and the agent’s confidence level. Usually a market of higher risk also brings a higher earning if the correct investment is made. Within a chosen market, there are good investments and bad investments. Thus, there may be several good decisions among the various markets at the same time. A similar situation also happens when one decides on the options for pension fund investments, for which choices such as growth, balanced, and stable funds are available. On the more entertaining side, betting on horse racing also provides choices of different levels of risk through different betting pools like win, place, quinella (i.e., betting on the first and second places not in order), etc. All of these situations involve the choice of several options, with a different payoff for each option.

To include multiple options into EMG, we consider a number N of agents where N is not a multiple of 3. Each agent chooses among three options, say, 1, 2, and 3, at each time step. The agents in the side with the least (most) number of agents, i.e., in the minority (majority) side, win (lose) and are awarded (deducted) one point, while the agents in the third option with intermediate number of agents get R point, where R may be negative or positive and $|R| < 1$. The outcome in increasing number of agents, i.e., increasing popularity, choosing the three options $(\Omega_1(t), \Omega_2(t), \Omega_3(t))$ at the time step t forms the publicly known information of the game. For example, the outcome (3,1,2) implies that option 3 (2) is the least (most) popular option. Every agent uses the set of outcomes of the most recent m time steps, $\mu(t) = \{(\Omega_1(t-m), \Omega_2(t-m), \Omega_3(t-m)), \dots, (\Omega_1(t-1), \Omega_2(t-1), \Omega_3(t-1))\}$, as the information to predict the current trend and the outcome at time step t . For given m , there are a total of $3^m \times 2^m$ different $\mu(t)$. The agents are also provided with a strategy that corresponds to the outcomes of the most recent occurrences of each of the $3^m \times 2^m$ possible μ , as in the original EMG [12]. This strategy gives the most recent trend, i.e., what happened in terms of popularity among the three options the last time that a particular μ occurred. The strategy is dynamical in the sense that it changes as the game proceeds.

At any time step of the game, each agent carries his own set of probabilities $(p_1, p_2, p_3 = 1 - (p_1 + p_2))$, with $0 \leq p_1 \leq 1$, $0 \leq p_2 \leq 1$, and $0 \leq (p_1 + p_2) \leq 1$. Given the most recent m outcomes, i.e., for given $\mu(t)$, an agent has the probability p_1 (p_3) to choose the predicted least (most) popular option as suggested by the strategy, and probability p_2 to follow the prediction on the intermediate option. Initially, the p value of each agent is assigned randomly, and the score is set to zero.

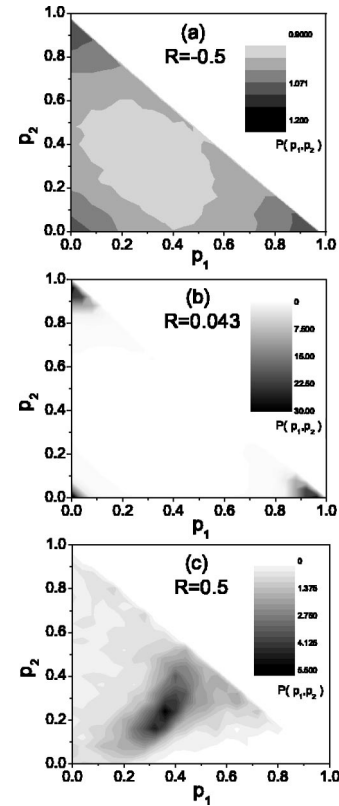


FIG. 1. The distribution $P(p_1, p_2)$ of the p values among the agents in a system of $N=1001$ with $r=0.2$, $m=3$, $d=-4$ for (a) $R = -0.5$, (b) 0.043 , and (c) 0.5 , projected on to the p_1 - p_2 . The grey scale indicates the value of $P(p_1, p_2)$.

As the game proceeds, the performance is registered in the score of the agents. If an agent’s score falls below a certain threshold value d ($d \leq 0$), he is allowed to replace his p values by new values of p_1 and p_2 within a range r centered at the current values p_1 and p_2 , and his score is reset to zero. Since $p_3 = 1 - p_1 - p_2$, it is sufficient to work in the p_1 - p_2 space. A reflective boundary condition [12] is imposed in the (p_1, p_2) space to ensure that the requirements $0 \leq p_i \leq 1$ ($i=1, 2, 3$) are satisfied. Therefore, evolution comes in by allowing agents to modify their p values. The present model is different from an earlier version of multiple-choice EMG proposed by Metzler and Horn [18] in that we allow for possibly more than one winning option. The model is also different from the previously proposed multiple-choice MG [19,20] in that the second least popular option may also be a winning choice and adaptability is introduced through the p values instead of different strategies assigned to the agents.

III. RESULTS

Extensive numerical simulations have been carried out to study the p -value distribution $P(p_1, p_2)$ in the population. Consider a system with $N=1001$ agents, $m=3$, $r=0.2$, and $d=-4$. Figure 1 shows typical $P(p_1, p_2)$ on a grey-scale two-dimensional plot projected on to the p_1 - p_2 plane for the different values of R ($R = -0.5, 0.043, 0.5$). Results are obtained by averaging over ten independent runs for each value of R ,

with an initially uniform distribution. In each run, the distribution is obtained in a time window of 10^5 time steps, after transient behavior dies off. In constructing $P(p_1, p_2)$, the p_1 and p_2 axes are each divided into 25 divisions, i.e., each division corresponds to 0.04. The distribution $P(p_1, p_2)$ is normalized so that $\iint P(p_1, p_2) dp_1 dp_2 = 1$. The distribution $P(p_1, p_2)$ for $R = -0.5$ is rather flat [see Fig. 1(a)], implying that there is no advantage in having any specific set of (p_1, p_2, p_3) over others. As R increases, the number of agents taking on extreme actions [$(p_1, p_2, p_3) \approx (1, 0, 0)$ or $(0, 1, 0)$ or $(0, 0, 1)$], i.e., persistently making a certain choice, increases, and there exists a small range of R in which $P(p_1, p_2) \approx 0$ for intermediate values of (p_1, p_2) [see Fig. 1(b)]. The distribution $P(p_1, p_2)$ is nearly symmetric about $(p_1 \approx 0.3, p_2 \approx 0.3)$ with peaks around $(p_1, p_2) = (0, 1)$, $(1, 0)$ and $(0, 0)$. This behavior is analogous to the segregation of agents into extreme actions in the original EMG [12–15]. The result implies that in order to flourish in such a population in situations for which $R \approx 0$, an agent should behave in an extreme way. This segregation behavior leads to an enhancement in the performance of the population as a whole in that the number of agents in the least popular option takes on a value approaching the limit $N/3$, as allowed by the definition of the winning minority side. Although results are shown only for the case of $m=3$ and $d=-4$, the steady-state distribution $P(p_1, p_2)$ does not depend sensitively on m and d , and the initial distribution. When R is further increased [see Fig. 1(c)], the game allows for more winners than losers. In this case, the agents become less decisive or cautious. The distribution shows a peak at about $(p_1 \approx 0.36, p_2 \approx 0.28)$, implying that the strategy's prediction of the winning option will be too crowded to win. It should be noted that the steady-state distribution in this case is dependent on the initial distribution. It is analogous to the freezing phenomena in the original EMG when the resource level is high [16]. Here, R plays the role of a resource level in that a positive R implies a majority of agents will earn a reward per turn. The lifespan $L(p_1, p_2)$, defined as the average duration for an agent holding (p_1, p_2) between modifications of p values, shows similar behavior as the distribution $P(p_1, p_2)$.

The performance of the system can be related to the average number of agents in each option and the fluctuations (variance) in the number of agents making a particular choice over time. Figure 2 shows the average fraction of agents (right axis) in the most (least) popular option and the third option and the variance σ^2/N in the number of agents (left axis) for different values of R ($m=3$, $d=-2$ and $r=0.2$), respectively. The average number of agents in each option takes on values close to $N/3$, with the winning option determined by a margin of about ten agents in a system with $N = 1001$. For $R < 0$, i.e., when there are more losers than winners, the fraction of agents in each option is insensitive to R . As R increases to the vicinity of $R \approx 0$, the average fraction of agents in the least (most) popular option reaches a maximum (minimum) at some value R_c with $R_c \approx 0$, signifying an enhanced performance of the system. Interestingly, the fraction of agents in the third (intermediate) option remains flat and close to $1/3$ over a wide range of R . As R is further

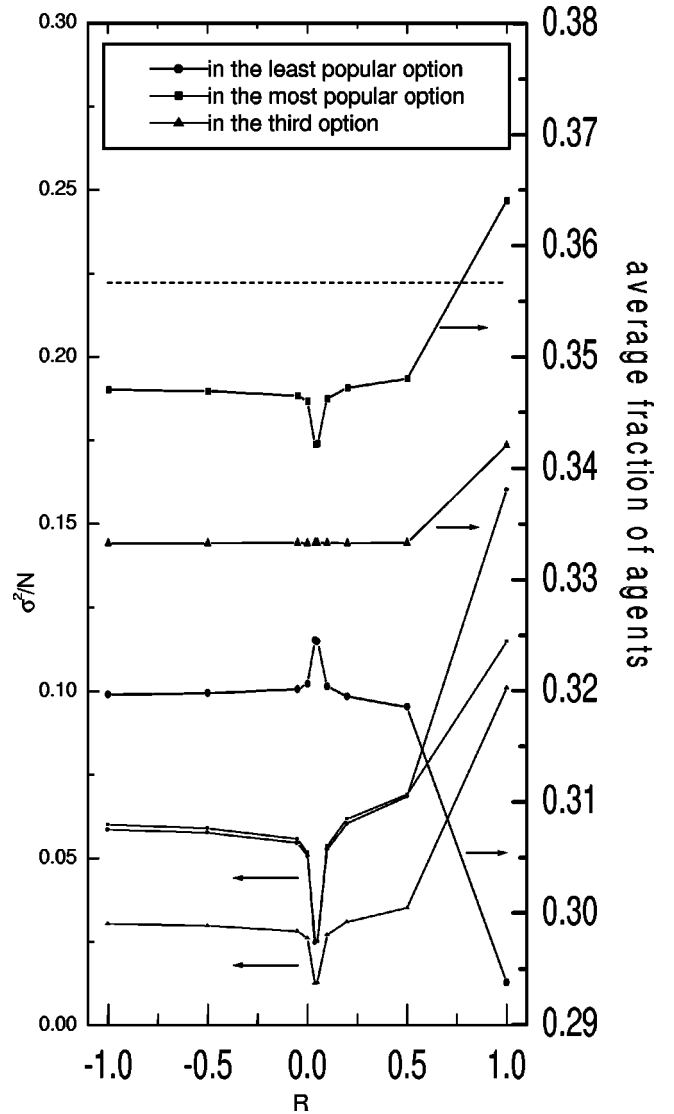


FIG. 2. The average fraction of agents (right axis) taking the three options (different symbols) and the corresponding variances σ^2/N (left axis) for different values of R . Other parameters are $m=3$, $d=-2$, and $r=0.2$. The lines are a guide for the eyes. The dashed line gives the variance in Eq. (1) corresponding to random decisions.

increased ($0 < R < 1$), more agents choose to follow the strategy's prediction on the least popular option and the predicted option becomes the most popular and hence too crowded to win. As a result, fewer agents choose the least popular (winning) option.

The dependence of the variance σ^2/N on R , for each of the options, is nonmonotonic and shows a corresponding drop and reaches a minimum at $R \approx R_c$. A smaller fluctuation implies a higher number of winners per turn, and hence better performance as a whole. The results imply an optimum cooperation in the population. For $R > R_c$, the variance increases with R . For comparison, the dashed line gives the variance σ_{rand}^2/N for random decisions in a multiple-choice game which is given by

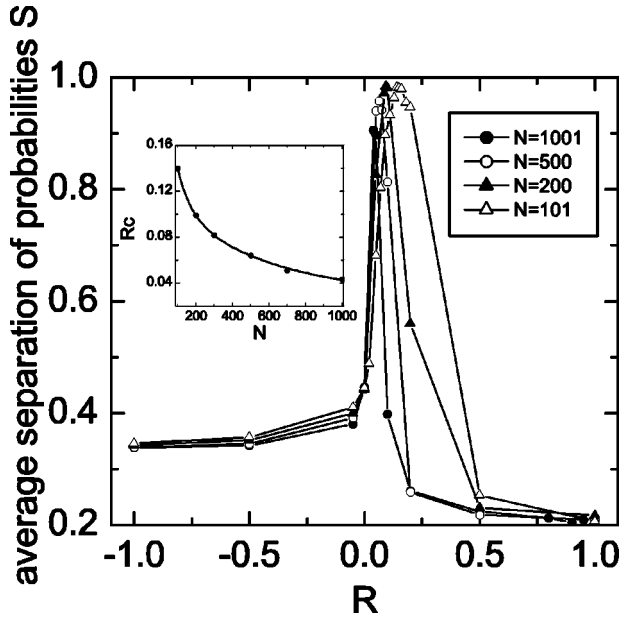


FIG. 3. The average separation of probabilities S among the agents for different number of agents $N=1001, 500, 200,$ and 101 in the system. Other parameters are $m=3, r=0.2, d=-4$. The separation S shows a maximum at $R_c(N)$. The inset shows the dependence of R_c on N .

$$\frac{\sigma_{\text{rand}}^2}{N} = \frac{1}{N} \sum_{m=0}^N q^{N-m} (1-q)^m C_N^m ((N-m) - Nq)^2, \quad (1)$$

where q is the probability of taking a side. For the three-option case, $q=1/3$, and $\sigma_{\text{rand}}^2/N=2/9$. It should be noted that the variance σ^2/N for all the values of R studied is smaller than σ_{rand}^2/N , implying that the agents are actually benefitted from the evolutionary nature of the game. Thus, R_c signifies an *optimal* reward (resource) of the system in that at $R=R_c$, a *maximum fraction* of the total possible reward per turn, e.g., $(N-2)/3 - (1-R)(1+(N-2)/3)$ for $N=1001$, is actually awarded to the agents.

To look closer into the change in the agents' behavior for different values of R , it is convenient to study the separation in the p values among the agents in the population. We define an averaged separation S by

$$S = \frac{2}{N(N-1)T} \sum_{t=1}^T \sum_{i,j=1}^N \sum_{\alpha=1,2,3} [p_{\alpha}(i) - p_{\alpha}(j)]^2, \quad (2)$$

where T is the time window for taking the average. The separation S is thus a mean over the difference squared of the p values of the agents. Segregation in p values among the agents corresponds to a larger value of S . Figure 3 shows the

average separation S for different population sizes $N=1001, 500, 200,$ and 101 ($m=3, r=0.2, d=-4$). For negative R , the fraction of agents taking extreme action is small, so the average reduced distance is small. As R increases, more agents show extreme behavior, resulting in a rapid increase in S with a maximum near $R=R_c$. The peak in S becomes narrower as the system size becomes larger. For $R>R_c$, the separation S drops quite rapidly and attains a value smaller than that of $R<0$. The results in Fig. 3 indicate that we may identify the value of $R_c(N)$ as the value of R at which the separation S attains a maximum. The resulting $R_c(N)$ (inset of Fig. 3) shows that R_c drops monotonically with N to a value of $R_c \approx 0.04$ for $N=1001$. The trend of the results proposes that R_c may approach the value of 0 as N is further increased.

IV. SUMMARY

We have proposed and studied a three-option EMG with different payoffs to the agents in each of the options. Interesting behavior, including self-segregation into extreme groups and nondecisiveness, are found, depending on the payoff to the third option (besides the minority and majority options). In difficult situations ($R<R_c$), segregation occurs and the agents prefer to take extreme action ($(p_1, p_2, p_3) \approx (1, 0, 0)$ or $(0, 1, 0)$ or $(0, 0, 1)$). For $R>R_c$, the p values of the agents tend to cluster at a common location. Detailed analysis on the average number of agents taking each option and the corresponding variance revealed an optimal performance of the system at some value R_c . Studies on systems of different sizes showed that R_c drop nearly to zero as the number of agents N in the system increases. A negative and positive value of R correspond to very different situations in that for $R<0$, there is a majority of losers in the population, while for $R>0$ the system allows for a majority of winners. The present model, employing a reflective boundary condition, represents a natural and straight forward generalization of the original two-option EMG [12]. It will be interesting to investigate the effects of different boundary conditions. However, we note that there is no natural and unique way to define, for example, a periodic boundary condition for multiple options. In addition, the feature that $R_c \rightarrow 0$ in the large population size limit in the present model is similar to that of $\bar{R}_c \rightarrow 1$ in Ref. [17]. Analytically, it is therefore possible to explain results of multiple-options EMG presented here along the line of the theory of the two-option EMG [21].

ACKNOWLEDGMENTS

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